

TORUS INVARIANT CURVES

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ABSTRACT. Using the language of T -varieties, we study torus invariant curves on a complete normal variety X with an effective codimension-one torus action. In the same way that the T -invariant Weil divisors on X are sums of “vertical” divisors and “horizontal” divisors, so too is each T -invariant curve a sum of “vertical” curves and “horizontal” curves. We give combinatorial formulas that calculate the intersection between T -invariant divisors and T -invariant curves, and generalize the celebrated toric cone theorem to the case of complete complexity-one T -varieties.

1. INTRODUCTION

A T -variety is a normal complex algebraic variety with an effective action of an algebraic torus. This definition matches the definition of a toric variety, except that the dimension of the torus may be less than the dimension of the variety on which it acts. In particular, any normal algebraic variety is a T -variety when endowed with the trivial action of $(\mathbb{C}^*)^0$. We therefore can’t expect to prove much about general T -varieties; we usually restrict our attention to *complexity-one* T -varieties, where the dimension of the torus is exactly one less than the dimension of the variety. In this paper, we study the T -invariant curves of a complete complexity-one T -variety, find formulas for their intersection with T -invariant divisors (using the theory of T -invariant divisors developed by Petersen and Süß in [PS]), and prove that the numerical equivalence classes of these curves generate the Mori cone of the T -variety.

We review the basics of T -varieties in Section 2. Informally speaking, a complexity-one T -variety is encoded by a family (parametrized by a projective curve Y) of polyhedral subdivisions of a vector space, all with the same tailfan. In Section 3, we describe two kinds of T -invariant curves in a T -variety, *vertical* curves and *horizontal* curves. The vertical curves correspond to walls (codimension-one strata) of one of these polyhedral subdivisions, while the horizontal curves correspond to certain maximal-dimensional cones of the tailfan. We give formulas that calculate the intersection of these curves with a T -invariant divisor using the language of Cartier support functions from [PS].

In Section 4, we generalize the toric cone theorem, which states that the Mori cone of a toric variety is generated as a cone by the classes of T -invariant curves corresponding to the walls of its fan. In our generalization, we show that the Mori cone of a complete complexity-one T -variety is generated as a cone by the classes of a finite collection of vertical curves and horizontal curves. We end the paper with examples in Section 5.

2. PRIMER ON T -VARIETIES

In this section, we review the basic notation and construction of T -varieties. The presentation favors brevity over pedagogy; we encourage any reader unfamiliar with T -varieties to read the excellent survey article [A] for a friendlier exposition to this beautiful topic.

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2.1. Notation. Let $T \cong (\mathbb{C}^*)^k$ be an algebraic torus, and M, N be the character lattice of T and the lattice of 1-parameter subgroups of T respectively. These lattices embed in the vector spaces

$$N_{\mathbb{Q}} := \mathbb{Q} \otimes N \quad M_{\mathbb{Q}} := \mathbb{Q} \otimes M$$

and are dual to one another¹. In classic toric geometry, one studies the correspondence between cones (and fans) in $N_{\mathbb{Q}}$ and the toric varieties encoded by these combinatorial data. Analogously, we study T -varieties through the correspondence between combinatorial gadgets called p -divisors (and divisorial fans) and the T -varieties they encode. Informally speaking, a p -divisor is a Cartier divisor on a semiprojective variety Y with polyhedral coefficients; a divisorial fan is a collection of p -divisors whose polyhedral coefficients “fit together” in a suitable way. To make formal these definitions, we begin by discussing monoids of polyhedra.

Let σ be a pointed cone in $N_{\mathbb{Q}}$, and $\sigma^{\vee} \subseteq M_{\mathbb{Q}}$ its dual. The set $\text{Pol}_{\mathbb{Q}}^+(N, \sigma)$ of all polyhedra in $N_{\mathbb{Q}}$ having σ as its tailcone (with the convention that $\emptyset \in \text{Pol}_{\mathbb{Q}}^+(N, \sigma)$) is a monoid under Minkowski addition with identity element σ . Any nonempty $\Delta \in \text{Pol}_{\mathbb{Q}}^+(N, \sigma)$ defines a map

$$(1) \quad \begin{aligned} h_{\Delta} : \sigma^{\vee} &\rightarrow \mathbb{Q} \\ u &\mapsto \min_{v \in \Delta} \langle v, u \rangle \end{aligned}$$

called the *support function* of Δ . A nonempty $\Delta \in \text{Pol}_{\mathbb{Q}}^+(N, \sigma)$ also defines a *normal quasifan* $\mathcal{N}(\Delta)$ in $M_{\mathbb{Q}}$ consisting of a cone λ_F for each face F of Δ defined by

$$\lambda_F = \{u \in \sigma^{\vee} \mid \langle u, v \rangle = h_{\Delta}(u) \ \forall v \in F\}.$$

The figure below shows an example of a polyhedron and its normal quasifan.

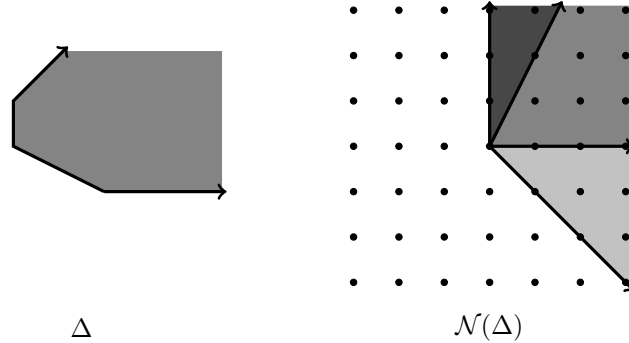


FIGURE 1. A polyhedron and its normal quasifan

Proposition 2.1. ([AH03], Lemma 1.4 and Proposition 1.5) *The support function h_{Δ} is a well-defined map whose regions of linearity are the maximal cones of $\mathcal{N}(\Delta)$. Moreover, any function in $\text{Hom}(\sigma^{\vee}, \mathbb{Q})$ whose regions of linearity define a quasifan can be realized as h_{Δ} for some Δ .*

Let $\text{Pol}_{\mathbb{Q}}(N, \sigma)$ be the Grothendieck group of $\text{Pol}_{\mathbb{Q}}^+(N, \sigma)$. Let Y be a semiprojective variety, with $\text{CaDiv}(Y)$ its group of Cartier divisors. An element

$$\mathcal{D} \in \text{Pol}_{\mathbb{Q}}(N, \sigma) \otimes_{\mathbb{Z}} \text{CaDiv}(Y)$$

is a *polyhedral divisor* with tailcone σ if it has a representative of the form $\mathcal{D} = \sum \mathcal{D}_P \otimes P$ for some $\mathcal{D}_P \in \text{Pol}_{\mathbb{Q}}^+(N, \sigma)$ and P prime². We will describe a procedure for constructing an affine T -variety

¹In this paper, when a picture of $N_{\mathbb{Q}}$ is juxtaposed with a picture of $M_{\mathbb{Q}}$, the reader may assume that the bases for these vector spaces have been chosen so that the pairing between them is the standard dot product.

²Because σ (not \emptyset) is the identity element of $\text{Pol}_{\mathbb{Q}}(N, \sigma)$, the summation notation in this sentence implies that only finitely many of the polyhedral coefficients \mathcal{D}_P differ from σ

from a certain kind of polyhedral divisor (called a *p-divisor*); this construction will involve taking the spectrum of the global sections of a sheaf of rings defined over a subset of Y . This subset, called the *locus* of \mathcal{D} , is

$$\text{Loc}(\mathcal{D}) := Y \setminus \bigcup_{\mathcal{D}_P = \emptyset} P.$$

The *evaluation* of \mathcal{D} at $u \in M \cap \sigma^\vee$ is the \mathbb{Q} -Cartier divisor³

$$\mathcal{D}(u) := \sum_{\mathcal{D}_P \neq \emptyset} h_{\mathcal{D}_P}(u) P|_{\text{Loc}(\mathcal{D})}.$$

We say that \mathcal{D} is a *p-divisor* if $\mathcal{D}(u)$ is semiample for all $u \in \sigma^\vee$ and big for all u in the interior of σ^\vee . The direct sum of the sheaves defined by the evaluations $\mathcal{D}(u)$ is an M -graded sheaf of rings

$$\mathcal{O}(\mathcal{D}) := \bigoplus_{u \in \sigma^\vee \cap M} \mathcal{O}_{\text{Loc}(\mathcal{D})}(\mathcal{D}(u)) \chi^u$$

over $\text{Loc}(\mathcal{D})$. There are two different T -varieties encoded by the p -divisor \mathcal{D}

$$\widetilde{TV}(\mathcal{D}) := \text{Spec}_{\text{Loc}(\mathcal{D})} \mathcal{O}(\mathcal{D}) \quad \text{and} \quad TV(\mathcal{D}) := \text{Spec } \Gamma(\text{Loc}(\mathcal{D}), \mathcal{O}(\mathcal{D}))$$

where the torus action is given by the M -grading on $\mathcal{O}(\mathcal{D})$. All affine T -varieties can be constructed this way.

Theorem 2.2. ([AH03], Corollary 8.14) *Every normal affine variety with an effective torus action can be realized as $TV(\mathcal{D})$ for some p -divisor \mathcal{D}*

Similar to the way that a fan of a non-affine toric variety can be obtained by “gluing together” the cones constituting an affine cover, so too can a non-affine T -variety be encoded by “gluing together” the p -divisors constituting an affine cover. To make formal these concepts, we first define the *intersection* of two p -divisors $\mathcal{D}, \mathcal{D}'$ on Y as the p -divisor

$$\mathcal{D} \cap \mathcal{D}' := \sum (\mathcal{D}_P \cap \mathcal{D}'_P) \otimes P.$$

We say that \mathcal{D}' is a *face* of \mathcal{D} if $\mathcal{D}'_P \subseteq \mathcal{D}_P$ for each P and the induced map $TV(\mathcal{D}') \rightarrow TV(\mathcal{D})$ is an open embedding. A finite set \mathcal{S} of p -divisors on Y is a *divisorial fan* if for any $\mathcal{D}, \mathcal{D}' \in \mathcal{S}$, $\mathcal{D} \cap \mathcal{D}'$ is an element of \mathcal{S} and is a face of both \mathcal{D} and \mathcal{D}' . We define $TV(\mathcal{S})$ and $\widetilde{TV}(\mathcal{S})$ to be the T -varieties obtained by gluing together the T -varieties $\{TV(\mathcal{D})\}_{\mathcal{D} \in \mathcal{S}}$ and $\{\widetilde{TV}(\mathcal{D})\}_{\mathcal{D} \in \mathcal{S}}$ according to these face relations. This process is detailed in [AHS08].

2.2. Geometry of $TV(\mathcal{S})$ and $\widetilde{TV}(\mathcal{S})$. Because $\widetilde{TV}(\mathcal{D})$ is defined as the relative spectrum of a sheaf of rings on $\text{Loc}(\mathcal{D})$, there is a natural projection map $\pi : \widetilde{TV}(\mathcal{D}) \rightarrow \text{Loc}(\mathcal{D}) \subseteq Y$. Because $TV(\mathcal{D})$ is defined as the spectrum of the global sections of the structure sheaf on $\widetilde{TV}(\mathcal{D})$, we also have a natural map $p : \widetilde{TV}(\mathcal{D}) \rightarrow \Gamma(\widetilde{TV}(\mathcal{D}), \mathcal{O}_{\widetilde{TV}(\mathcal{D})}) \cong TV(\mathcal{D})$. Given a divisorial fan \mathcal{S} , the maps π, p corresponding to the different $\mathcal{D} \in \mathcal{S}$ glue into maps

$$\begin{array}{ccc} \widetilde{TV}(\mathcal{S}) & \xrightarrow{p} & TV(\mathcal{S}) \\ \downarrow \pi & & \\ Y & & \end{array}$$

In this subsection, we describe the fibers of p and π . In particular, we will notice that for $y \in Y$, the reduced fiber $\pi^{-1}(y)$ is a union of irreducible toric varieties, and that the contraction map p identifies

³Some authors define a “ \mathbb{Q} -Cartier” divisor to be a Weil divisor with a Cartier multiple. Our \mathbb{Q} -Cartier divisors are elements of $\mathbb{Q} \otimes \text{Div}(Y)$ having a Cartier multiple (so may have rational coefficients). The pedantic reader is invited to replace all instances of “ \mathbb{Q} -Cartier divisor” in this paper with “ \mathbb{Q} -Cartier \mathbb{Q} -divisor”.

certain disjoint torus orbits of $\widetilde{TV}(\mathcal{S})$. Many of these results simplify when $TV(\mathcal{S})$ is a complexity-one T -variety; because this is the only case we will need for later sections, we will henceforth assume that Y is a projective curve. The reader interested in higher-complexity T -varieties should read [A] for the more general results.

In [P], the author describes the reduced fibers of π using the language of *dappled toric bouquets*. We begin by reviewing this language.

Definition 2.3. The *fan ring* of a quasifan Λ in $M_{\mathbb{Q}}$ is

$$\mathbb{C}[\Lambda] := \bigoplus_{u \in |\Lambda| \cap M} \mathbb{C}\chi^u$$

with multiplication defined by

$$\chi^u \chi^v = \begin{cases} \chi^{u+v} & \text{if } u, v \in \lambda \text{ for some cone } \lambda \in \Lambda \\ 0 & \text{otherwise} \end{cases}$$

For a nonempty $\Delta \in \text{Pol}_{\mathbb{Q}}^+(N, \sigma)$ and a cone λ_F of its inner normal quasifan $\mathcal{N}(\Delta)$, let

$$M_{\lambda_F} := \{u \in \lambda_F \cap M \mid h_{\Delta}(u) \in \mathbb{Z}\}.$$

Remark 2.4. In other papers, M_{λ_F} is defined differently: when $\Delta \otimes [P]$ appears as a summand in a p -divisor, the elements $u \in M_{\lambda_F}$ are required to satisfy the condition that $h_{\Delta}(u)[P]$ is locally principal at P . In the complexity-one case, this condition coincides with our condition that $h_{\Delta}(u) \in \mathbb{Z}$.

Finally, let $S_{\Delta} \subseteq |\Lambda(\Delta)| \cap M$ consist of those u such that $S_{\Delta} \cap \lambda_F = M_{\lambda_F}$ for every cone $\lambda_F \in \mathcal{N}(\Delta)$. S_{Δ} can be thought of as a conewise-varying sublattice of M . The figure below shows an example of S_{Δ} for a given Δ ; the elements of $S_{\Delta} \subseteq M$ are in bold.

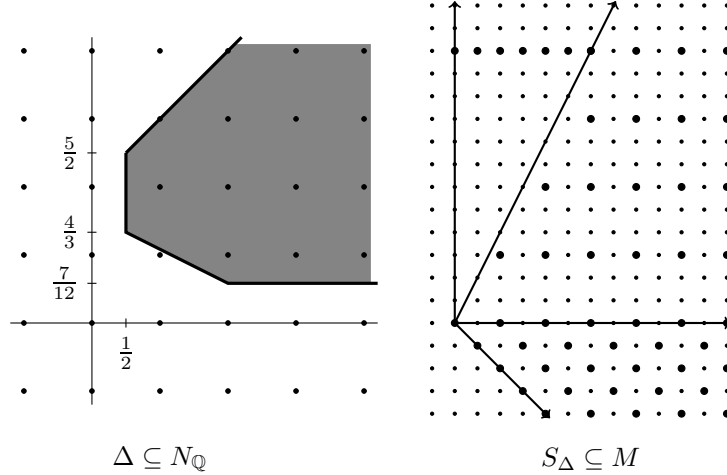


FIGURE 2. S_{Δ} is a conewise-varying sublattice of M

Definition 2.5. The *dappled fan ring* of Δ is the following subring of $\mathbb{C}[\mathcal{N}(\Delta)]$

$$\mathbb{C}[\mathcal{N}(\Delta), S_{\Delta}] := \bigoplus_{u \in S_{\Delta}} \mathbb{C}\chi^u$$

Definition 2.6. The *dappled toric bouquet* encoded by Δ is the variety $TB(\Delta) := \text{Spec}(\mathbb{C}[\mathcal{N}(\Delta), S_{\Delta}])$. Given a polyhedral complex $\Sigma = \{\Delta\}$ in $N_{\mathbb{Q}}$, the dappled toric bouquet encoded by Σ is the variety $TB(\Sigma)$ obtained by gluing the $\{TB(\Delta)\}_{\Delta \in \Sigma}$ according to the face relations among the polyhedra.

Observe that $TB(\Delta)$ and $TB(\Sigma)$ have a natural torus action induced by the M -grading of the dappled fan rings. For a T -variety $TV(\mathcal{S})$ over Y and a point $y \in Y$, the polyhedra $\{\mathcal{D}_y\}_{\mathcal{D} \in \mathcal{S}}$ fit together into a polyhedral complex \mathcal{S}_y of $N_{\mathbb{Q}}$.

Proposition 2.7. *[[P], Prop 1.29] Let \mathcal{S} be a p -divisor on the smooth projective curve Y . The reduced fiber $\pi^{-1}(y)$ of $\pi : \widetilde{TV}(\mathcal{S}) \rightarrow Y$ is equivariantly isomorphic to $TB(\mathcal{S}_y)$.*

Motivated by Proposition 2.7 to study the geometry of non-affine toric bouquets, we construct a fan for each vertex of a polyhedral subdivision Σ of $N_{\mathbb{Q}}$; the toric varieties they encode will be precisely the irreducible components of $TB(\Sigma)$. For a vertex $v \in \Sigma$, define the lattice

$$M_v = \{u \in M \mid \langle u, v \rangle \in \mathbb{Z}\}$$

Because M_v is a sublattice of M , N is a sublattice of $N_v := M_v^{\vee} \subseteq N_{\mathbb{Q}}$. Let $i_v : N_{\mathbb{Q}} \rightarrow (N_v)_{\mathbb{Q}}$ be the map induced by this inclusion. As Δ ranges over all polyhedra in Σ containing v , the cones $i_v(\mathbb{Q}_{\geq 0} \cdot (\Delta - v))$ form a fan F_v in $(N_v)_{\mathbb{Q}}$. For any cone $\sigma = i_v(\mathbb{Q}_{\geq 0} \cdot (\Delta - v))$ of F_v , the semigroup $\sigma^{\vee} \cap N_v^{\vee}$ is isomorphic to the semigroup $\lambda_{\Delta} \cap S_{\Delta}$. Because this isomorphism commutes with the gluing data induced by the face relations, we have the following description of the irreducible components of $TB(\Sigma)$.

Proposition 2.8. *The irreducible components of $TB(\Sigma)$ are equivariantly isomorphic to the toric varieties $\{TV(F_v)\}$ where the set ranges over the vertices v of Σ .*

For example, the polyhedral complex in Figure 3 encodes a toric bouquet with three irreducible toric components. We have drawn the lattices N_v not as a square grid, but in a way that the sublattice $N \subseteq N_v$ (in bold) is a square grid so that the angles between the polyhedra are preserved. In the example, one fan encodes \mathbb{P}^2 and the other fans encode weighted projective spaces.

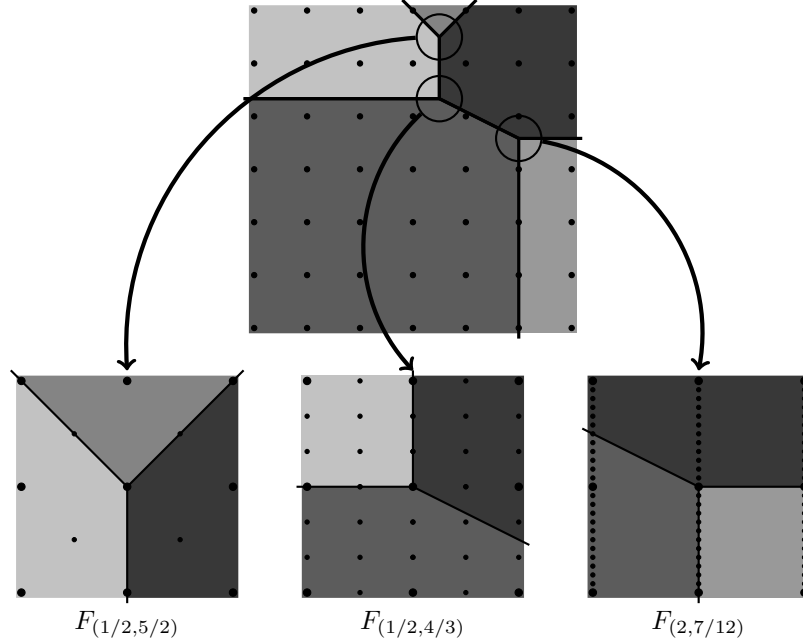


FIGURE 3. Components of a toric bouquet

Given a divisorial fan \mathcal{S} , its *tailfan* $\text{tail}(\mathcal{S})$ is the fan consisting of the tailcones of the p -divisors comprising \mathcal{S} . Because the coefficients \mathcal{D}_y of a p -divisor \mathcal{D} differ from its tailcone for only finitely

many y , the polyhedral subdivisions \mathcal{S}_y differ from $\text{tail}(\mathcal{S})$ for only finitely many y . By Proposition 2.8, the fiber of π over $y \in Y$ is equal to $TV(\text{tail}(\mathcal{S}))$ for all but finitely many y and specializes to a (possibly non-reduced) union of toric varieties at these finitely many points.

By the discussion above, the familiar orbit-cone correspondence for toric varieties translates into a correspondence between T -orbits in $\widetilde{TV}(\mathcal{S})$ and pairs (y, F) where $y \in Y$ and $F \in \mathcal{S}_y$. To understand $TV(\mathcal{S})$, we will describe how the map p identifies certain of these orbits in different fibers. We first consider the case of an affine T -variety. For a p -divisor \mathcal{D} with tailcone σ and a $u \in \sigma^\vee \cap M$, the semiample divisor $\mathcal{D}(u)$ defines a map

$$\xi_u : \text{Loc}(\mathcal{D}) \rightarrow \text{Proj} \left(\bigoplus_{k \geq 0} \Gamma(\text{Loc}(\mathcal{D}), \mathcal{D}(ku)) \right).$$

Theorem 2.9. ([AH03], Theorem 10.1) *The map $p : \widetilde{TV}(\mathcal{D}) \rightarrow TV(\mathcal{D})$ induces a surjection*

$$\{(y, F) : y \in Y, F \text{ is a face of } \mathcal{D}_y\} \rightarrow \{T\text{-orbits in } TV(\mathcal{D})\}$$

that identifies the orbits corresponding to (y, F) and (y', F') iff $\lambda_F = \lambda_{F'} \subseteq M_{\mathbb{Q}}$ and $\xi_u(y) = \xi_u(y')$ for some (equivalently, for any) $u \in \text{relint}(\lambda_F)$.

In the non-affine case, the gluing maps among $\{TV(\mathcal{D})\}_{\mathcal{D} \in \mathcal{S}}$ are prescribed by the face relations between the p -divisors, which identifies precisely those T -orbits in $TV(\mathcal{D})$ and $TV(\mathcal{D}')$ corresponding to the faces $\{(y, \mathcal{D}_y \cap \mathcal{D}'_y)\}_{y \in Y}$.

3. T -INVARIANT CURVES AND INTERSECTION THEORY

In this section, we study the intersection theory of complete complexity-one T -varieties over a projective curve Y . **For the rest of the paper, all T -varieties are complete, complexity-one T -varieties over a projective curve Y .** The “completeness” condition translates into the combinatorial requirement that $|\mathcal{S}_y| = N_{\mathbb{Q}}$ for all y . Motivated by the correspondence between T -invariant Cartier divisors and Cartier support functions introduced in [PS], we define the notion of a \mathbb{Q} -Cartier support function to encode \mathbb{Q} -Cartier torus invariant divisors. We will describe two kinds of T -invariant curves – *vertical* curves and *horizontal* curves – then give formulas that compute the intersection of these curves with a T -invariant \mathbb{Q} -Cartier divisor.

Definition 3.1. Given a nontrivial $\Delta \in \text{Pol}_{\mathbb{Q}}^+(N, \sigma)$ and an affine $\varphi : \Delta \rightarrow \mathbb{Q}$, the *linear part* of φ is the function

$$\begin{aligned} \text{lin} \varphi : \sigma &\rightarrow \mathbb{Q} \\ n &\mapsto \varphi(p + n) - \varphi(p) \end{aligned}$$

where p is any point in Δ . If $\langle \sigma \rangle \subseteq N_{\mathbb{Q}}$ is the subspace spanned by σ , the function $\text{lin} \varphi$ extends uniquely to a linear function $\langle \sigma \rangle \rightarrow \mathbb{Q}$, which will also be written $\text{lin} \varphi$ without risk of confusion.

Definition 3.2. Let \mathcal{S} be the divisorial fan of a complexity-one T -variety over Y . A \mathbb{Q} -Cartier support function is a collection of affine functions

$$\{h_{\mathcal{D}, y} : |\mathcal{D}_y| \rightarrow \mathbb{Q}\}_{\substack{\mathcal{D} \in \mathcal{S} \\ y \in Y}}$$

with rational slope and rational translation such that

- (1) For a fixed $y \in Y$, the functions $\{h_{\mathcal{D}, y}\}_{\mathcal{D} \in \mathcal{S}}$ define a continuous function $h_y : |\mathcal{S}_y| \rightarrow \mathbb{Q}$. That is, $h_{\mathcal{D}, y}$ and $h_{\mathcal{D}', y}$ agree on $\mathcal{D}_y \cap \mathcal{D}'_y$ for $\mathcal{D}, \mathcal{D}' \in \mathcal{S}$.
- (2) For each $\mathcal{D} \in \mathcal{S}$ with complete locus, there exists $u \in M, f \in K(Y)$ and $N \in \mathbb{Z}_{>0}$ such that $Nh_{\mathcal{D}, y}(v) = -\text{ord}_y(f) - \langle u, v \rangle$ for all $y \in Y$ and all $v \in N_{\mathbb{Q}}$.

- (3) If $\mathcal{D}_y, \mathcal{D}'_y$ have the same tailcone, then $\text{lin} h_{\mathcal{D},y} = \text{lin} h_{\mathcal{D}',y}$.
- (4) For a fixed \mathcal{D} , $h_{\mathcal{D},y}$ differs from $\text{lin} h_{\mathcal{D},y}$ for only finitely many y .

A \mathbb{Q} -Cartier support function is called a *Cartier support function* if each $h_{\mathcal{D},y}$ has integral slope and integral translation and $N = 1$ in condition (2). We write $\text{CaSF}(\mathcal{S})$ and $\mathbb{Q}\text{CaSF}(\mathcal{S})$ to denote the abelian group (under standard addition of functions) of Cartier support functions and \mathbb{Q} -Cartier support functions respectively.

For any T -invariant Cartier divisor D on $TV(\mathcal{S})$ and any p -divisor $\mathcal{D} \in \mathcal{S}$, we can always find an open cover $\{U_i\}$ of Y for which there exists Cartier data for $D|_{TV(\mathcal{D})}$ of the form $(TV(\mathcal{D})|_{U_i}, f_i \chi^{u_i})$ (see proof of [PS], Prop 3.10 for details). These Cartier data define functions

$$\{h_{\mathcal{D},y}(v) = -\text{ord}_y(f_i) - \langle u_i, v \rangle\}_{y \in U_i}$$

which agree on the overlaps of the U_i to define $h_{\mathcal{D},y}$ for all y . In this way, we can define a Cartier support function for any Cartier divisor on $TV(\mathcal{S})$.

Proposition 3.3. ([PS], Prop 3.10) *Let $T - \text{CaDiv}(\mathcal{S})$ denote the group of T -invariant Cartier divisors on $TV(\mathcal{S})$. This association of a Cartier support function to a T -invariant Cartier divisor defines an isomorphism of groups*

$$T - \text{CaDiv}(\mathcal{S}) \cong \text{CaSF}(\mathcal{S})$$

If $\{h_{\mathcal{D},y}\}$ is the Cartier support function for ND , where $N > 0$ and D is a T -invariant \mathbb{Q} -Cartier divisor, then $\{N^{-1}h_{\mathcal{D},y}\}$ is a \mathbb{Q} -Cartier support function. In this way, we can associate a \mathbb{Q} -Cartier support function to any T -invariant \mathbb{Q} -Cartier divisor on $TV(\mathcal{S})$. The following is an immediate corollary of Proposition 3.3.

Corollary 3.4. *Let $T - \mathbb{Q}\text{CaDiv}(\mathcal{S})$ denote the group of T -invariant \mathbb{Q} -Cartier divisors on $TV(\mathcal{S})$. Then the association described above is an isomorphism of groups*

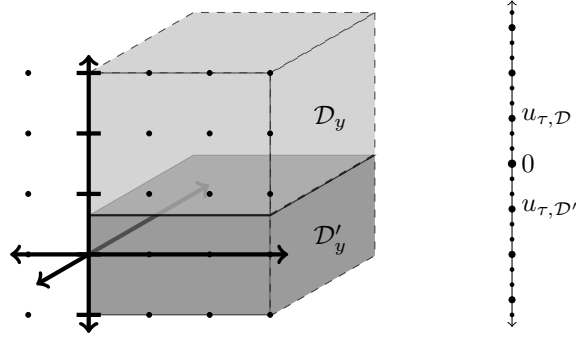
$$T - \mathbb{Q}\text{CaDiv}(\mathcal{S}) \cong \mathbb{Q}\text{CaSF}(\mathcal{S})$$

3.1. Vertical Curves. Toward the goal of describing the intersection theory of a T -variety, we study its T -invariant curves. We start with *vertical curves*, which are images (under p) of a T -invariant curve contained in a single fiber of π .

Recall from Proposition 2.8 that for $y \in Y$, the reduced fiber $\pi^{-1}(y)$ has as its irreducible components a toric variety for each vertex v of \mathcal{S}_y . A toric variety has a T -invariant curve corresponding to each wall⁴ of its fan (by taking the closure of the corresponding torus orbit). Translating this fact into the context of toric bouquets, we call a codimension-one element of a polyhedral complex a *wall* if it can be realized as the intersection of two top-dimensional polyhedra; there is a T -invariant curve in a toric bouquet for each wall of the corresponding polyhedral complex. In this section, we study the curves in $\widehat{TV}(\mathcal{S})$ and $TV(\mathcal{S})$ corresponding to these T -invariant curves.

Fix a T -variety $TV(\mathcal{S})$ and a point $y \in Y$. Let $\tau \in \mathcal{S}_y$ be a wall of the polyhedral complex \mathcal{S}_y , let $\mathcal{D}, \mathcal{D}' \in \mathcal{S}$ be two p -divisors for which $\tau = \mathcal{D}_y \cap \mathcal{D}'_y$, let $\lambda_{\tau,\mathcal{D}} \subseteq M_{\mathbb{Q}}$ be the cone in $\mathcal{N}(\mathcal{D}_y)$ dual to τ , and let $u_{\tau,\mathcal{D}}$ be the semigroup generator of $M_{\lambda_{\tau,\mathcal{D}}}$. As usual, unweildy notation obfuscates a simple picture: if \mathcal{D} and \mathcal{D}' have polyhedral coefficients over y as shown in Figure 4 (τ is the horizontal plane in a single orthant at a height of $2/3$), then the sublattice $\mathbb{Z} \cdot u_{\tau,\mathcal{D}} = M_{\lambda_{\tau,\mathcal{D}}} \cup M_{\lambda_{\tau,\mathcal{D}'}}$ consists of the bold elements of the vertical axis of $M \cong \mathbb{Z}^3$ shown on the right.

⁴A *wall* of a fan is a codimension-one cone that can be realized as the intersection of two top-dimensional cones.

FIGURE 4. The sublattice $\mathbb{Z} \cdot u_{\tau, \mathcal{D}}$ corresponding to the wall $\tau = \mathcal{D}_y \cap \mathcal{D}'_y$

If $g \in K(Y)$ is Cartier data for $\mathcal{D}(u_{\tau, \mathcal{D}})$ in some neighborhood of y , the maps

$$\begin{aligned} \Gamma(\text{Loc}(\mathcal{D}), \mathcal{O}(\mathcal{D})) &\rightarrow \mathbb{C}[z] \\ f\chi^u &\mapsto \begin{cases} 0 & u \notin M_{\lambda_\tau, \mathcal{D}} \\ (g^k f)(y)z^k & u = ku_{\tau, \mathcal{D}} \end{cases} \\ \Gamma(\text{Loc}(\mathcal{D}'), \mathcal{O}(\mathcal{D}')) &\rightarrow \mathbb{C}[z^{-1}] \\ f\chi^u &\mapsto \begin{cases} 0 & u \notin M_{\lambda_\tau, \mathcal{D}'} \\ (g^{-k} f)(y)z^{-k} & u = -ku_{\tau, \mathcal{D}} \end{cases} \end{aligned}$$

glue together to induce a map

$$(2) \quad \mathbb{P}^1 \rightarrow TV(\mathcal{S})$$

the image of which we will call the *vertical curve* $C_{\tau, y}$.

Proposition 3.5. *The vertical curve $C_{\tau, y}$ is the image under p of the closure of the torus orbit in $TB(\mathcal{S}_y) \subseteq \widetilde{TV}(\mathcal{S})$ corresponding to the wall τ .*

Proof. For any affine open $U \subseteq Y$ containing y , Map 2 factors

$$(3) \quad \mathbb{P}^1 \rightarrow \pi^{-1}(U) \subseteq \widetilde{TV}(\mathcal{S}) \xrightarrow{p} TV(\mathcal{S})$$

where $\mathbb{P}^1 \rightarrow \pi^{-1}(U)$ is given by

$$\begin{aligned} (4) \quad \Gamma(U, \mathcal{O}(\mathcal{D})) &\rightarrow \mathbb{C}[z] \\ f\chi^u &\mapsto \begin{cases} 0 & u \notin M_{\lambda_\tau, \mathcal{D}} \\ (g^k f)(y)z^k & u = ku_{\tau, \mathcal{D}} \end{cases} \\ \Gamma(U, \mathcal{O}(\mathcal{D}')) &\rightarrow \mathbb{C}[z^{-1}] \\ f\chi^u &\mapsto \begin{cases} 0 & u \notin M_{\lambda_\tau, \mathcal{D}'} \\ (g^{-k} f)(y)z^{-k} & u = -ku_{\tau, \mathcal{D}} \end{cases} \end{aligned}$$

Therefore, it suffices to show that the image of $\mathbb{P}^1 \rightarrow \pi^{-1}(U)$ is the closure of the torus orbit in $TB(\mathcal{S}_y) \subseteq \widetilde{TV}(\mathcal{S})$ corresponding to the wall τ . To do so, we recall some relevant details about the isomorphism between the reduced fibers of π and a dappled toric bouquet (see [AH03], Proposition 7.10 for details). This isomorphism is constructed by first choosing a collection of functions $\{g_{\mathcal{D}(u)} \in K(Y)\}_{u \in S_\Delta}$ such that, after possibly shrinking U ,

$$\text{div}(g_{\mathcal{D}(u)})|_U = \mathcal{D}(u)|_U \quad \text{and} \quad g_{\mathcal{D}(u+u')} = g_{\mathcal{D}(u)}g_{\mathcal{D}(u')}$$

Then the isomorphism between the fiber and the dappled toric bouquet is induced by

$$(5) \quad \begin{aligned} \Gamma(U, \mathcal{O}(\mathcal{D})) &\rightarrow \mathbb{C}[\mathcal{N}(\mathcal{D}_y), S_{\mathcal{D}_y}] \\ f\chi^u &\mapsto \begin{cases} (g_{\mathcal{D}(u)}f)(y)\chi^u & \text{if } u \in S_{\mathcal{D}_y} \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

(and similarly for \mathcal{D}'). On the other hand, the closure of the torus orbit corresponding to τ in the toric bouquet is parametrized by gluing the maps

$$(6) \quad \begin{aligned} \mathbb{C}[\mathcal{N}(\mathcal{D}_y), S_{\mathcal{D}_y}] &\rightarrow \mathbb{C}[z] \\ \chi^u &\mapsto \begin{cases} z^k & \text{if } u = ku_{\tau, \mathcal{D}}, k \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases} \\ \mathbb{C}[\mathcal{N}(\mathcal{D}'_y), S_{\mathcal{D}'_y}] &\rightarrow \mathbb{C}[z^{-1}] \\ \chi^u &\mapsto \begin{cases} z^{-k} & \text{if } u = -ku_{\tau, \mathcal{D}}, k \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

The composition of Equation 5 and Equation 6 yields Equation 4, proving the claim. \square

To find a formula that calculates the intersection between $C_{\tau, y}$ and a T -invariant Cartier divisor D , we pick Cartier data for D that includes two sets of the form

$$\{(TV(\mathcal{D}|_U), f\chi^u), (TV(\mathcal{D}'|_{U'}), f'\chi^{u'})\}$$

where $U, U' \subseteq Y$ are open sets containing y . The Cartier support function for D includes

$$h_{\mathcal{D}, y} = -\text{ord}_y(f) - \langle u, v \rangle \quad \text{and} \quad h_{\mathcal{D}', y} = -\text{ord}_y(f') - \langle u', v \rangle.$$

Because $h_{\mathcal{D}, y}$ and $h_{\mathcal{D}', y}$ agree on τ , it must be the case that

$$\text{ord}_y(f) - \text{ord}_y(f') + \langle u - u', v \rangle = 0$$

for all $v \in \tau$. In particular, $u - u' \in \mathbb{Q} \cdot u_{\tau, \mathcal{D}}$. Moreover, since $\langle u - u', v \rangle = \text{ord}_y(f') - \text{ord}_y(f) \in \mathbb{Z}$, it must be the case that $\langle u - u', v \rangle \in \mathbb{Z}$ for $v \in \tau$. Therefore, $u - u' = ku_{\tau, \mathcal{D}}$ for some $k \in \mathbb{Z}$, and the quotient of the two Cartier data is $f\chi^u/f'\chi^{u'} = (f/f')\chi^{ku_{\tau, \mathcal{D}}}$. Under the parametrization of $C_{\tau, y}$ in Equation 2, this rational function pulls back to $(g^k f/f')(y)z^k$ on $C_{\tau, y} \cong \mathbb{P}^1$, where g is Cartier data for $\mathcal{D}(u_{\tau, \mathcal{D}})$. Therefore, the degree of the pullback of D onto $C_{\tau, y}$ is k . This is precisely $\mu_{\tau}^{-1}\langle u - u', n_{\tau, \mathcal{D}} \rangle$, where μ_{τ} is the index of $\mathbb{Z} \cdot u_{\tau, \mathcal{D}}$ in $\mathbb{Q} \cdot u_{\tau, \mathcal{D}} \cap M$ and $n_{\tau, \mathcal{D}} \in N$ is any representative of the generator of $N/(u_{\tau, \mathcal{D}})^{\perp}$ that pairs positively with $u_{\tau, \mathcal{D}}$ (equivalently, $n_{\tau, \mathcal{D}}$ is any element of N such that $\langle n_{\tau, \mathcal{D}}, u_{\tau, \mathcal{D}} \rangle = \mu_{\tau}$).

$$\langle D, C_{\tau, y} \rangle = \mu_{\tau}^{-1}\langle u - u', n_{\tau, \mathcal{D}} \rangle$$

or, using the language of Cartier support functions,

$$(7) \quad \langle D, C_{\tau, y} \rangle = \mu_{\tau}^{-1}(\text{lin}h_{\mathcal{D}', y} - \text{lin}h_{\mathcal{D}, y})(n_{\tau, \mathcal{D}})$$

By linearity, the same formula applies when D is a T -invariant \mathbb{Q} -Cartier divisor.

Example 3.6. Let $\mathcal{D}, \mathcal{D}' \in \mathcal{S}$ be p -divisors that have the slices shown in Figure 4. Suppose that with respect to the standard basis given by the coordinate axes in the picture, a T -invariant divisor D has the following Cartier support functions

$$h_{\mathcal{D}, y}(v) = -10 + \langle (9, 4, 17), v \rangle \quad \text{and} \quad h_{\mathcal{D}', y}(v) = 0 + \langle (9, 4, 2), v \rangle$$

Then

$$\langle D, C_{\tau, y} \rangle = 3^{-1}\langle (0, 0, -15), (0, 0, 1) \rangle = -5$$

3.2. Horizontal Curves. Let σ be a full-dimensional cone of $\text{tail}(\mathcal{S})$. Because the T -varieties we study are complete, every \mathcal{S}_y contains a polyhedron with tailcone σ . Such a polyhedron corresponds to a fixed point in the fiber $\pi^{-1}(y)$. Taking the union (as y varies) of these fixed points defines a curve $\tilde{C}_\sigma \subseteq \widetilde{TV}(\mathcal{S})$. Theorem 2.9 shows that p contracts \tilde{C}_σ precisely if there is some $\mathcal{D} \in \mathcal{S}$ with tailcone σ and complete locus. In this case, we say that σ is *marked*.

Definition 3.7. A cone σ of $\text{tail}(\mathcal{S})$ is *marked* if σ is the tailcone of a p -divisor $\mathcal{D} \in \mathcal{S}$ with complete locus.

When σ is unmarked, Theorem 2.9 shows that no distinct points of \tilde{C}_σ are identified by p . Toward the goal of finding an intersection formula for these *horizontal curves* $C_\sigma := p(\tilde{C}_\sigma)$, we parametrize them. Let $TV(\mathcal{S})$ be a T -variety and let σ be an unmarked full-dimensional cone of $\text{tail}(\mathcal{S})$. For $\mathcal{D} \in \mathcal{S}$ with tailcone σ , we have a map of rings

$$(8) \quad \begin{aligned} \varphi_{\mathcal{D}} : \Gamma(TV(\mathcal{D}), \mathcal{O}(\mathcal{D})) &\rightarrow \Gamma(\text{Loc}(\mathcal{D}), \mathcal{O}_Y) \\ f\chi^u &\mapsto \begin{cases} f & \text{if } u = 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Because each $\{\text{Loc}(\mathcal{D}) \mid \text{tail}(\mathcal{D}) = \sigma\}$ is affine, these glue into a map

$$s_\sigma : Y \hookrightarrow TV(\mathcal{S})$$

where we used the fact that \mathcal{S} is complete (so $|\mathcal{S}_y| = N_{\mathbb{Q}}$ for all y) to deduce that Y is covered by $\{\text{Loc}(\mathcal{D}) \mid \text{tail}(\mathcal{D}) = \sigma\}$. The map s_σ factors through $\widetilde{TV}(\mathcal{S})$. By carefully following the isomorphism between the fibers of π and the corresponding toric bouquets (as in the proof of Proposition 3.5), we see that the image of s_σ indeed equals the horizontal curve $p(\tilde{C}_\sigma)$.

We can use this parametrization to find an intersection formula for T -invariant divisors and horizontal curves. Fix a cone σ of $\text{tail}(\mathcal{S})$ of full dimension and a T -invariant Cartier divisor D with Cartier support function $\{h_{\mathcal{D},y}\}$. Because σ has full dimension, there is a unique $u_\sigma \in M$ and collection of integers $\{a_y \in \mathbb{Z}\}_{y \in Y}$ such that for each \mathcal{D} with tailcone σ and each $y \in \text{Loc}(\mathcal{D})$,

$$h_{y,\mathcal{D}}(v) = -a_y - \langle u_\sigma, v \rangle.$$

We can find Cartier data for D whose open sets and rational functions are of the form

$$(TV(\mathcal{D})|_U, f_{\mathcal{D},U}\chi^{u_\sigma})$$

for open sets $U \subseteq Y$. Then $\text{ord}_y(f_{\mathcal{D},U}) = a_y$ for all \mathcal{D} with tailcone σ and $y \in U$. When σ is unmarked, the open sets U appearing in the Cartier data are affine, and the pullback of the transition function $f_{\mathcal{D},U}f_{\mathcal{D}',U'}^{-1}\chi^0$ onto the curve $C_\sigma \cong Y$ is the function $f_{\mathcal{D},U}f_{\mathcal{D}',U'}^{-1}$ on $U \cap U'$. That is, the functions $f_{\mathcal{D},U}$ appearing in the Cartier data for D are themselves the Cartier data for the pullback of D onto $C_\sigma \cong Y$. As a Weil divisor, the pullback of D onto C_σ is $\sum a_y[y]$; we call this divisor D_σ .

Definition 3.8. Given a \mathbb{Q} -Cartier support function $\{h_{\mathcal{D},y}\}$, a cone σ of full dimension in $\text{tail}(\mathcal{S})$, and a point y , there is a unique $a_y \in \mathbb{Z}$ such that for every \mathcal{D} with tailcone σ and $\text{Loc}(\mathcal{D}) \ni y$,

$$h_{\mathcal{D},y} = -a_y - \text{lin}(h_{\mathcal{D},y}).$$

Then define

$$D_\sigma = \sum_{y \in Y} a_y[y]$$

Remark 3.9. The definition of D_σ makes sense even when σ is marked. However, if \mathcal{D} has complete locus, then by ([PS], Proposition 3.1) every invariant Cartier divisor on $TV(\mathcal{D})$ is principal. It follows that $\deg(D_\sigma) = 0$ for every marked σ .

Remark 3.10. Compare this definition to ([PS], Definition 3.26). In our notation, $D_\sigma = -h|_\sigma(0)$.

With this new definition, we can summarize the discussion above with the following equation for the intersection theory of a T -invariant divisor with a horizontal curve.

$$\langle D, C_\sigma \rangle = \deg(D_\sigma)$$

By linearity, the same formula applies when D is a T -invariant \mathbb{Q} -Cartier divisor.

4. THE T CONE THEOREM

Given a normal variety X , let $Z_1(X)$ be the proper 1-cycles, and define

$$N^1(X) := (\text{CaDiv}(X)/\sim) \otimes_{\mathbb{Z}} \mathbb{R} \quad N_1(X) := (Z_1(X)/\sim) \otimes_{\mathbb{Z}} \mathbb{R}$$

where \sim denotes numerical equivalence of divisors in the first definition, and numerical equivalence of curves in the second. The vector space $N^1(X)$ contains the cone $\text{Nef}(X)$ generated by classes of nef divisors, and the vector space $N_1(X)$ contains the cone $NE(X)$ generated by classes of irreducible complete curves. The *Mori cone* $\overline{NE}(X)$ is the closure of $NE(X)$. With respect to the intersection product, $N_1(X)$ and $N^1(X)$ are dual vector spaces, and the cones $\text{Nef}(X)$, $\overline{NE}(X)$ are dual cones.

When X is the toric variety of a fan Σ , the closure of the torus orbit corresponding to a wall of Σ defines an element of $\overline{NE}(X)$. The celebrated toric cone theorem ([CLO], Theorem 6.3.20(b)) states that $\overline{NE}(X)$ is generated as a cone by these classes. In this section, we prove the corresponding result for T -varieties. We continue to assume that all T -varieties are complete complexity-one T -varieties over a projective curve Y .

Theorem 4.1. *Let $TV(\mathcal{S})$ be an n -dimensional T -variety, and let $y' \in Y$ be any point for which $\mathcal{S}_{y'} = \text{tail}(\mathcal{S})$. Then*

$$(9) \quad \overline{NE}(TV(\mathcal{S})) = \sum_{\substack{y \in Y \\ \tau \text{ a wall of } \mathcal{S}_y \\ \dim(\text{tail}(\tau)) < n-1}} \mathbb{R}_{\geq 0}[C_{\tau,y}] + \sum_{\substack{\tau \text{ a wall} \\ \text{of } \text{tail}(\mathcal{S})}} \mathbb{R}_{\geq 0}[C_{\tau,y'}] + \sum_{\substack{\sigma \in \text{tail}(\mathcal{S}) \\ \dim(\sigma) = n-1 \\ \sigma \text{ unmarked}}} \mathbb{R}_{\geq 0}[C_\sigma]$$

For the proof, we review two important facts about divisors on T -varieties.

Proposition 4.2. *Any Cartier divisor D on a T -variety $TV(\mathcal{S})$ is linearly equivalent to a T -invariant Cartier divisor.*

Different authors have different definitions of concavity; to us, a function $\varphi : N_{\mathbb{Q}} \rightarrow \mathbb{Q}$ is concave if $\varphi(tv + (1-t)w) \geq t\varphi(v) + (1-t)\varphi(w)$ for all $v, w \in N_{\mathbb{Q}}$ and all $t \in [0, 1]$

Proposition 4.3. ([PS], Corollary 3.29) *A T -invariant Cartier divisor $D \in T - \text{CaDiv}(\mathcal{S})$ with Cartier support function $\{h_{\mathcal{D},y}\}$ is nef iff all h_y are concave and $\deg(D_\sigma) \geq 0$ for every maximal cone σ of the tailfan.*

Toward our goal of proving Theorem 4.1, we will use Proposition 4.3 to show that a Cartier divisor is nef if it intersects all vertical and horizontal curves nonnegatively. The proof of this fact requires a combinatorial lemma. Given a Cartier support function $\{h_{\mathcal{D},y}\}$ and any $\mathcal{D} \in \mathcal{S}$, $y \in Y$ such that $\dim(\mathcal{D}_y) = \dim(N_{\mathbb{Q}})$, define $\tilde{h}_{\mathcal{D},y} : N_{\mathbb{Q}} \rightarrow \mathbb{Q}$ to be the unique affine function that extends $h_{\mathcal{D},y} : |\mathcal{D}_y| \rightarrow \mathbb{Q}$.

Lemma 4.4. *Let $\{h_{\mathcal{D},y}\}$ be a Cartier support function. The following are equivalent*

- $h_y : N_{\mathbb{Q}} \rightarrow \mathbb{Q}$ is concave.
- For every wall $\tau = \mathcal{D}_y \cap \mathcal{D}'_y$ of \mathcal{S}_y , there is some $v \in \mathcal{D}'_y \setminus \mathcal{D}_y$ with $h_{\mathcal{D}',y}(v) \leq \tilde{h}_{\mathcal{D},y}(v)$.

Proof. This is a straightforward extension of ([CLO], Lemma 6.1.5 (a) \iff (d)) (where it is proved for Cartier support functions on a fan). \square

Proposition 4.5. *A Cartier divisor $D \in \text{CaDiv}(TV(\mathcal{S}))$ is nef iff $\langle D, C \rangle \geq 0$ for all vertical and horizontal curves C .*

Proof. The forward direction follows from the definition of nef. To prove the reverse direction, let $D \in \text{CaDiv}(\mathcal{S})$ satisfy the condition that $\langle D, C \rangle \geq 0$ for all vertical and horizontal curves. Replace D with a linearly equivalent T -invariant divisor and let $\{h_{\mathcal{D},y}\}$ be its Cartier support function. Let $\tau = \mathcal{D}_y \cap \mathcal{D}'_y$ be a wall of \mathcal{S}_y . Fix any $n_{\tau,\mathcal{D}} \in N$ with $\langle n_{\tau,\mathcal{D}}, u_{\tau,\mathcal{D}} \rangle = \mu_{\tau}$ and any $v_{\tau} \in \text{relint}(\tau)$. Then pick $\epsilon > 0$ such that $v := v_{\tau} + \epsilon n_{\tau,\mathcal{D}} \in \mathcal{D}_y \setminus \mathcal{D}'_y$. Then

$$\begin{aligned} h_{\mathcal{D},y}(v) &= h_{\mathcal{D},y}(v_{\tau}) + \text{lin} h_{\mathcal{D},y}(\epsilon n_{\tau,\mathcal{D}}) \\ \tilde{h}_{\mathcal{D}',y}(v) &= h_{\mathcal{D}',y}(v_{\tau}) + \text{lin} h_{\mathcal{D}',y}(\epsilon n_{\tau,\mathcal{D}}). \end{aligned}$$

Because $h_{\mathcal{D},y}$ and $h_{\mathcal{D}',y}$ agree on τ ,

$$\tilde{h}_{\mathcal{D}',y}(v) - h_{\mathcal{D},y}(v) = (\text{lin} h_{\mathcal{D}',y} - \text{lin} h_{\mathcal{D},y})(\epsilon n_{\tau,\mathcal{D}}) \geq 0$$

where the final inequality comes from applying Equation 7 to the fact that $\langle D, C_{\tau,y} \rangle \geq 0$. Because this holds for all walls in all slices \mathcal{S}_y , we conclude by Lemma 4.4 that each h_y is concave.

To show that $\deg(D_{\sigma}) \geq 0$ for every maximal cone σ of the tailfan, observe that if σ is marked, then $\deg(D_{\sigma}) = 0$ by Remark 3.9; if σ is unmarked, then $\deg(D_{\sigma}) = \langle D, C_{\sigma} \rangle \geq 0$. \square

To put Proposition 4.5 in context, remember that a T -variety has infinitely many distinct vertical curves. Indeed, if τ is a wall of $\text{tail}(\mathcal{S})$, then for every $y \in Y$ there is (by completeness) a vertical curve $C_{\tau',y}$ where τ' is a wall of \mathcal{S}_y with tailcone τ . The next proposition shows that the classes of all such curves lie on a single ray of $N_1(TV(\mathcal{S}))$.

Proposition 4.6. *Let $\tau = \sigma \cap \sigma'$ be a wall of $\text{tail}(\mathcal{S})$, where σ, σ' are full dimensional cones of $\text{tail}(\mathcal{S})$. The classes*

$$\mathcal{C}_{\tau} = \left\{ [C_{\tau',y}] \mid \begin{array}{l} \tau' = \mathcal{D}_y \cap \mathcal{D}'_y \text{ for some } \mathcal{D}, \mathcal{D}' \text{ with} \\ \text{tail}(\mathcal{D}) = \sigma, \text{tail}(\mathcal{D}') = \sigma' \end{array} \right\} \subseteq N_1(TV(\mathcal{S}))$$

are positive multiples of each other. Specifically, $[C_{\tau_1,y_1}] = \mu_{\tau_1}^{-1} \mu_{\tau_2} [C_{\tau_2,y_2}]$.

Proof. Let $\{h_{\mathcal{D},y}\}$ be the Cartier support function of some $D \in \text{T-CaDiv}(TV(\mathcal{S}))$. All $h_{\mathcal{D},y}$ with $\text{tail}(\mathcal{D}) = \sigma$ (respectively σ') will have the same linear part, say $-u_{\sigma} \in M_{\mathbb{Q}}$ (respectively $-u_{\sigma'} \in M_{\mathbb{Q}}$). Then for two classes $[C_{\tau_1,y_1}], [C_{\tau_2,y_2}] \in \mathcal{C}_{\tau}$, Equation 7 calculates the intersections as

$$\langle D, C_{\tau_1,y_1} \rangle = \mu_{\tau_1}^{-1} \langle u_{\sigma} - u_{\sigma'}, n_{\tau_1,\mathcal{D}} \rangle \quad \langle D, C_{\tau_2,y_2} \rangle = \mu_{\tau_2}^{-1} \langle u_{\sigma} - u_{\sigma'}, n_{\tau_2,\mathcal{D}} \rangle$$

Since we can choose $n_{\tau_1,\mathcal{D}} = n_{\tau_2,\mathcal{D}}$, it follows that $\langle D, C_{\tau_1,y_1} \rangle = \mu_{\tau_1}^{-1} \mu_{\tau_2} \langle D, C_{\tau_2,y_2} \rangle$ for all D . \square

We are finally ready to prove Theorem 4.1. Using the propositions above, the proof is nearly identical to the proof of the toric cone theorem in ([CLO], Theorem 6.3.20(b)).

Proof. (Theorem 4.1) Let Γ be the rational polyhedral cone in $NE(TV(\mathcal{S}))$ defined by the right hand side of Equation 9. By definition, Γ includes the classes of all horizontal curves; by Proposition 4.6, it also includes the classes of all vertical curves. Therefore, Proposition 4.5 implies that $\Gamma^{\vee} = \text{Nef}(TV(\mathcal{S}))$, so $\Gamma = \Gamma^{\vee\vee} = \overline{NE}(TV(\mathcal{S}))$. \square

5. EXAMPLES

5.1. Example 1. Consider the divisorial fan \mathcal{S} shown in Figure 5. $TV(\mathcal{S})$ is the projectivized cotangent bundle of the first Hirzebruch surface. All horizontal divisors⁵ in $TV(\mathcal{S})$ are contracted. For each vertical divisor $D_{[y],v}$ and each maximal p -divisor $\mathcal{D}_i \in \mathcal{S}$, we write the Weil divisor $\sum a_y[y]$ and an element $u \in M$ in Table 1 to encode the Cartier support function $\{h_{\mathcal{D}_i,y}(w) = -a_y - \langle u, w \rangle\}$ of $D_{[y],v}$. For example, the Cartier support function for $D_{[0],(0,0)}$ includes $h_{\mathcal{D}_4,\infty}(v) = 1 - \langle (-2, -1), v \rangle$.

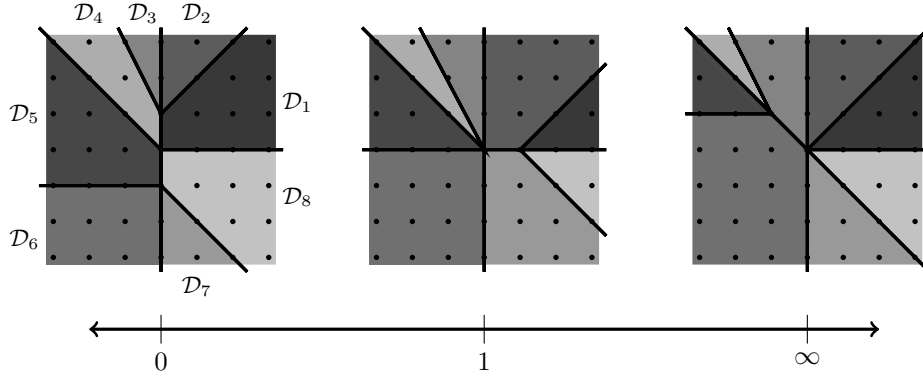


FIGURE 5.

	\mathcal{D}_1	\mathcal{D}_2	\mathcal{D}_3	\mathcal{D}_4	\mathcal{D}_5	\mathcal{D}_6	\mathcal{D}_7	\mathcal{D}_8
$D_{[0],(0,1)}$	0 (0, 1)	0 (0, 1)	0 (1, 1)	0 (1, 1)	0 (0, 0)	0 (0, 0)	0 (0, 0)	0 (0, 0)
$D_{[0],(0,0)}$	[0]–[1] (1, –1)	0 (0, 0)	0 (0, 0)	[0]–[∞] (–2, –1)	[0]–[∞] (0, 1)	0 (0, 0)	0 (0, 0)	[0]–[1] (1, 1)
$D_{[0],(0,-1)}$	0 (0, 0)	0 (0, 0)	0 (0, 0)	0 (0, 0)	0 (–1, –1)	0 (–1, –1)	0 (0, –1)	0 (0, –1)
$D_{[1],(1,0)}$	0 (1, 0)	0 (1, 0)	0 (0, 0)	0 (0, 0)	0 (0, 0)	0 (0, 0)	0 (1, 0)	0 (1, 0)
$D_{[1],(0,0)}$	0 (0, 0)	[1]–[0] (–1, 1)	[1]–[0] (1, 1)	[1]–[∞] (–1, 0)	[1]–[∞] (–1, 0)	[1]–[0] (–1, –1)	[1]–[0] (–1, –1)	0 (0, 0)
$D_{[∞],(-1,1)}$	0 (0, 0)	0 (0, 0)	0 (–1, 0)	0 (–1, 0)	0 (–1, 0)	0 (–1, 0)	0 (0, 0)	0 (0, 0)
$D_{[∞],(0,0)}$	[∞]–[1] (1, 0)	[∞]–[0] (0, 1)	[∞]–[0] (2, 1)	0 (0, 0)	0 (0, 0)	[∞]–[0] (0, –1)	[∞]–[0] (0, –1)	[∞]–[1] (1, 0)

TABLE 1. Torus invariant divisors on $TV(\mathcal{S})$

Because every maximal-dimensional cone of $\text{tail}(\mathcal{S})$ is marked, $TV(\mathcal{S})$ has no horizontal curves. Let $\tau_{i,j,y}$ be the wall of \mathcal{S}_y realized as the intersection between \mathcal{D}_i and \mathcal{D}_j (if such a wall exists). Using Proposition 4.6, we see that the numerical equivalence class of $C_{\tau_{i,j,y}}$ only depends on i and j ; to save space, we abbreviate $[C_{\tau_{i,j,y}}]$ as $C_{i,j}$.

As an example of a calculation, consider $C_{1,2}$ and the T -invariant divisor $D_{[0],(0,0)}$ with Cartier support function $\{h_{\mathcal{D}_y}\}$. Using notation from Section 3.1, $n_{\tau,\mathcal{D}_2} = (0, 1)$. The relevant linear parts

⁵See [PS] for a definition and description of horizontal and vertical divisors

of the Cartier support function are $\text{lin}h_{\mathcal{D}_{1,0}} = -(1, -1) \in M$ and $\text{lin}h_{\mathcal{D}_{2,0}} = (0, 0) \in M$. The intersection can then be calculated using Equation 7

$$\langle D_{[0],(0,0)}, C_{1,2} \rangle = 1^{-1} \langle (-1, 1) - (0, 0), (0, 1) \rangle = 1$$

The complete list of intersections is in Table 2. The canonical divisor is also listed; it can be expressed as a sum of the vertical divisors using the formula from ([PS], Theorem 3.21).

	$C_{1,2}$	$C_{2,3}$	$C_{3,4}$	$C_{4,5}$	$C_{5,6}$	$C_{6,7}$	$C_{7,8}$	$C_{8,1}$	$C_{1,4}$	$C_{5,8}$	$C_{2,7}$	$C_{3,6}$
$D_{[0],(0,1)}$	0	-1	0	1	0	0	0	1	-1	0	1	1
$D_{[0],(0,0)}$	1	0	1	-2	1	0	1	-2	3	1	0	0
$D_{[0],(0,-1)}$	0	0	0	1	0	1	0	1	0	1	1	1
$D_{[1],(1,0)}$	0	1	0	0	0	1	0	0	1	1	0	0
$D_{[1],(0,0)}$	1	-2	1	0	1	0	1	0	1	1	2	2
$D_{[\infty],(-1,1)}$	0	1	0	0	0	1	0	0	1	1	0	0
$D_{[\infty],(0,0)}$	1	-2	1	0	1	0	1	0	1	1	2	2
K_X	-2	2	-2	0	-2	-2	-2	0	-4	-4	-4	-4

TABLE 2. Intersections of divisors and curves on $TV(\mathcal{S})$

5.2. **Example 2.** Let $\sigma_1, \sigma_2, \sigma_3$ be the cones

$$\sigma_1 = \mathbb{Q}_{\geq 0} \cdot (1, 0) + \mathbb{Q}_{\geq 0} \cdot (0, 1)$$

$$\sigma_2 = \mathbb{Q}_{\geq 0} \cdot (0, 1) + \mathbb{Q}_{\geq 0} \cdot (-1, -1)$$

$$\sigma_3 = \mathbb{Q}_{\geq 0} \cdot (1, 0) + \mathbb{Q}_{\geq 0} \cdot (-1, -1)$$

and let \mathcal{S} be the divisorial fan on \mathbb{P}^1 having the following maximal p -divisors

$$\mathcal{D}_1 = ((2/3, 1/2) + \sigma_1)[0] + ((-2/3, -1/2) + \sigma_1)[1] + \emptyset[\infty]$$

$$\mathcal{D}_2 = ((2/3, 1/2) + \sigma_2)[0] + ((-2/3, -1/2) + \sigma_2)[1] + ((-1, -1) + \sigma_2)[\infty]$$

$$\mathcal{D}_3 = ((2/3, 1/2) + \sigma_3)[0] + ((-2/3, -1/2) + \sigma_3)[1] + ((-1, -1) + \sigma_3)[\infty]$$

$$\mathcal{D}_4 = \emptyset[0] + \emptyset[1] + ((-1, -1) + \sigma_1)[\infty]$$

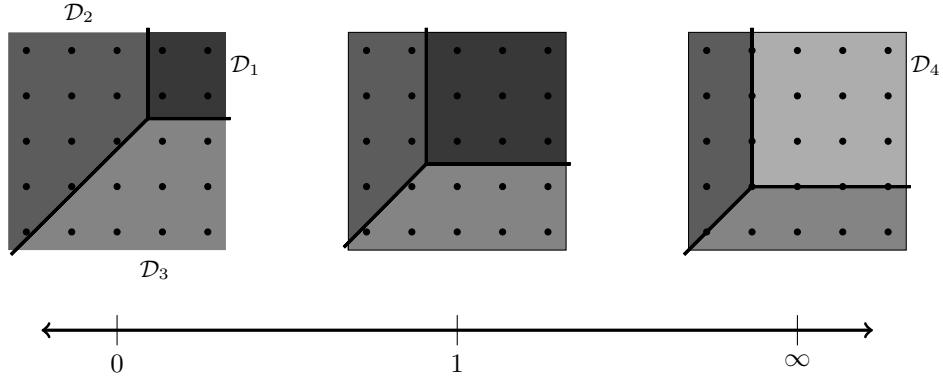


FIGURE 6. The divisorial fan \mathcal{S}

The T -variety corresponding to \mathcal{S} is a deformation of \mathbb{P}^3 . The T -invariant divisors and their intersections with T -invariant curves are encoded in Table 3 and 4 respectively, using the same notation as in the previous example.

	\mathcal{D}_1	\mathcal{D}_2	\mathcal{D}_3	\mathcal{D}_4
$D_{[0],(2/3,1/2)}$	$\begin{smallmatrix} 1/6[0] \\ (0,0) \end{smallmatrix}$	$\begin{smallmatrix} 5/18[0] - 1/9[1] - 1/6[\infty] \\ (-1/6,0) \end{smallmatrix}$	$\begin{smallmatrix} 1/4[0] - 1/12[1] - 1/6[\infty] \\ (0,-1/6) \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ (0,0) \end{smallmatrix}$
$D_{[1],(-2/3,-1/2)}$	$\begin{smallmatrix} 1/6[1] \\ (0,0) \end{smallmatrix}$	$\begin{smallmatrix} 1/9[0] + 1/18[1] - 1/6[\infty] \\ (-1/6,0) \end{smallmatrix}$	$\begin{smallmatrix} 1/12[0] + 1/12[1] - 1/6[\infty] \\ (0,-1/6) \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ (0,0) \end{smallmatrix}$
$D_{[\infty],(-1,-1)}$	$\begin{smallmatrix} 0 \\ (0,0) \end{smallmatrix}$	$\begin{smallmatrix} 2/3[0] - 2/3[1] \\ (-1,0) \end{smallmatrix}$	$\begin{smallmatrix} 1/2[0] - 1/2[1] \\ (0,-1) \end{smallmatrix}$	$\begin{smallmatrix} [\infty] \\ (0,0) \end{smallmatrix}$
$D_{\mathbb{Q}_{\geq 0} \cdot (1,0)}$	$\begin{smallmatrix} -2/3[0] + 2/3[1] \\ (0,0) \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ (-1/6,0) \end{smallmatrix}$	$\begin{smallmatrix} -1/6[0] + 1/6[1] \\ (0,-1/6) \end{smallmatrix}$	$\begin{smallmatrix} [\infty] \\ (0,0) \end{smallmatrix}$
$D_{\mathbb{Q}_{\geq 0} \cdot (0,1)}$	$\begin{smallmatrix} -1/2[0] + 1/2[1] \\ (0,1) \end{smallmatrix}$	$\begin{smallmatrix} 1/6[0] - 1/6[1] \\ (-1,1) \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ (0,0) \end{smallmatrix}$	$\begin{smallmatrix} [\infty] \\ (0,1) \end{smallmatrix}$

TABLE 3. Torus invariant divisors on $TV(\mathcal{S})$

	$C_{\tau_{1,2}}$	$C_{\tau_{2,3}}$	$C_{\tau_{1,3}}$	C_{σ_1}
$D_{[0],(2/3,1/2)}$	$1/6$	$1/6$	$1/6$	$1/6$
$D_{[1],(-2/3,-1/2)}$	$1/6$	$1/6$	$1/6$	$1/6$
$D_{[\infty],(-1,-1)}$	1	1	1	1
$D_{\mathbb{Q}_{>0} \cdot (1,0)}$	1	1	1	1
$D_{\mathbb{Q}_{>0} \cdot (0,1)}$	1	1	1	1

TABLE 4. Intersections on $TV(\mathcal{S})$

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